

comment on "Two Fermi points in the exact solution of the Kondo problem"

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In a recent paper [1] A. Zvyagin reexamined the solution of the Kondo model, and claimed that the model should be considered with two "Fermi points" in the presence of a magnetic field, and that a new energy scale arises as a result. I show below that these claims are in error.

The Kondo hamiltonian,

$$H = -i \int dx \psi_a^\dagger(x) \partial \psi_a(x) + J \psi_a^\dagger(0) \sigma_{ab}^i \psi_b(0)$$

describes a magnetic impurity in a metal, capturing the radial dynamics of a 3d model. The spectrum is linearized around the Fermi point (there is only 1 point since the Fermi surface in 3d is simply connected) and as a result the electrons are chiral - right movers by our convention - with a spectrum $E = p$.

The linearization is valid for J small and for momenta small compared to a cutoff D , of the order of the bandwidth. The cutoff is considered large compared to any physical quantity and the only scale remaining in the problem is the Kondo temperature $T_K = D \exp^{-\frac{\pi}{2J}}$. The results are universal and parameterized by $T_K \ll D$. If the coupling is not small the spectrum cannot be linearized and the universality is lost.

The model is soluble and its solution is given by the set of equations:

$$N^e \theta_1(\lambda_\alpha - 1) + \theta_1(\lambda_\alpha) = 2\pi J_\alpha + \sum_{\beta=1}^M \theta_2(\lambda_\alpha - \lambda_\beta) \quad (1)$$

where $\theta_n(\lambda) = -2 \tan^{-1}(2\lambda/cn)$, $c = 2J/(1 - 3/4J^2)$, N^e is the number of electrons interacting with a spin 1/2 impurity. There is a total of $N = N^e + 1$ spins (including the impurity), M of which are "down" and $N - M$ are "up". The spin of the state is $S = \frac{1}{2}(N - 2M)$. The M integers (or half integers) J_α are the quantum numbers of the states and each allowed configuration of them determines the state uniquely. From equation (1) we find that the allowed values of M quantum numbers J_α are between $J_{max} = (N - M - 1)/2$ and $J_{min} = -J_{max}$. Having solved the equations for a *distinct* set of λ 's the energy of the state is given by

$$E = D \sum_{\alpha=1}^M [\theta_1(\lambda_\alpha - 1) - \pi] + \sum_{j=1}^{N^e} \frac{2\pi}{L} n_j$$

where L is the size of the system, $D = N^e/L$ is the electron density (also serving as a cut off) and n_j are integers, the charge quantum numbers of the system.

The ground state of the system is a singlet $S = 0$ with the $M = N/2$ quantum numbers J_α occupying consecutively all $N/2$ slots between $J_{max} = (N/2 - 1)/2$ and $J_{min} = -(N/2 - 1)/2$. In the thermodynamic limit $N, M \rightarrow \infty$ and also λ_α takes values from $-\infty$ to ∞ . We shall denote the density of solutions by $\sigma(\lambda)$. Excited states correspond to other choices of allowed configurations. The simplest is the triplet, with $M = N/2 - 1$, leaving two "holes" in the sequence of quantum numbers between $J_{max} = (N/2)$ and $J_{min} = -J_{max}$. Each of these holes describes a spin-1/2 spinon with energy [2],

$$E^{spinon} = 2D \tan^{-1} \left[\exp \frac{\pi}{c} (\lambda^h - 1) \right] \quad (2)$$

where λ^h is the spin-rapidity of the spinon, corresponding to the unfilled slot J^h . The hole can take any value in the allowed range and correspondingly λ^h can take any real value in the thermodynamic limit. Note that for λ^h large and negative the energy is arbitrarily small while for λ^h large and positive it is very large, of the order of the cutoff. This asymmetry of the spinon spectrum is inherited from the asymmetry inherent in the linear spectrum of the right moving electrons. The spectrum of the spinons becomes linear in the momentum when the cutoff is removed [6]. (This asymmetry is in clear contradiction to statements in the last paragraph of the first column on page 060405 of ref. [1].)

A typical excited state will have holes (namely spinons) and may also have "strings" (complex λ 's) combining the spinons to lower spin states. For example, in addition to the two spinon triplet, there is a two spinon singlet, degenerate in energy with the triplet in the infinite volume limit, consisting of two holes and a 2-string.

We now turn on a magnetic field h at zero temperature, adding to the hamiltonian the term

$$H_{mag} = -2S h, \quad (3)$$

where S is the total spin component of the system in the direction of h . Note that H_{mag} commutes with the hamiltonian.

As a result of the magnetic term, the system will gain energy by flipping spins to align with the magnetic field. Each flipped spin corresponds to two holes in the λ sea, two spinons. The excitation energy of each spinon is given by Eq.(2), and can be made arbitrarily close to zero by choosing λ^h sufficiently large and negative. Hence the

magnetic field will excite spinons with λ large and negative. As all λ rapidities must be distinct, we are led to a depletion region where no λ solutions exist from $\lambda = -\infty$ to $\lambda = B$.

The “magnetic Fermi point” B is determined by an equilibrium argument equating the magnetic energy gained by flipping spins to align with the magnetic field and the energy cost of these holes. One finds

$$B = B(h) = \frac{c}{\pi} \ln\left(\frac{h}{T_h}\right)$$

With T_h a magnetic Kondo scale. The main point we wish to emphasize is that due to the form of the spinon excitation spectrum there is only one depletion region, not two as claimed by Zvyagin.

It is obvious that no complex strings are excited, as these reduce the total spin of the system. For example, consider the fundamental singlet and triplet excitations. While their cost in interaction energy is the same, the singlet does not gain magnetic energy. We thus have to consider the following magnetization equation determining the lowest energy state in the presence of a magnetic field [2,4,5]:

$$\sigma_B(\lambda) + \int_B^\infty K(\lambda - \lambda') \sigma_B(\lambda') d\lambda' = f_{kondo}(\lambda), \quad (4)$$

where,

$$K(\lambda) = \frac{1}{\pi} \frac{c}{c^2 + \lambda^2} = \frac{1}{\pi} \frac{d}{d\lambda} \theta(\lambda)$$

$$f_{kondo}(\lambda) = \frac{N^e}{\pi} \frac{(c/2)}{(c/2)^2 + (\lambda - 1)^2} + \frac{1}{\pi} \frac{(c/2)}{(c/2)^2 + \lambda^2}.$$

The energy and total spin of the system are given by

$$E(h) = E_{B(h)} = D \int_B^\infty \sigma_B(\lambda) [\theta_1(\lambda - 1) - \pi] + \sum_j \frac{2\pi}{L} n_j - 2h S \quad (5)$$

and

$$S = \frac{1}{2}N - M = \frac{1}{2}N - \int_B^\infty \sigma_B(\lambda) d\lambda. \quad (6)$$

The quantum numbers n_j have no spin content and are not excited by the magnetic field.

Equation (4) is a Wiener-Hopf integral equation and has been discussed using this technique by Yang and Yang in connection with a similar problem in the Heisenberg model [8]. It is amusing to point out that while Eq. (4) is exact in the case of the Kondo model, it is only approximate in the Heisenberg case. The reason is simple. The excitation spectrum for the Heisenberg model,

$$E^{spinon} = \frac{2J}{\cosh(\frac{\pi}{2}\lambda h)}$$

so that holes will be excited in the low energy regions $(-\infty, -B]$ and $[B, +\infty)$, and solutions λ_α will fall into the segment $[-B, B]$. Hence the magnetization equation takes the form [8],

$$\sigma_B(\lambda) + \int_{-B}^B K(\lambda - \lambda') \sigma_B(\lambda') d\lambda' = f_{heis}(\lambda),$$

where

$$f_{heis}(\lambda) = \frac{N}{\pi} \frac{(c/2)}{(c/2)^2 + (\lambda)^2}$$

As this equation is not of the Wiener Hopf form Yang and Yang develop a method to take into account both depletion regions for small magnetic fields (large B).

As another example consider the closely related Gross-Neveu model. The spinon spectrum is,

$$E^{spinon} = m \cosh\left(\frac{\pi}{c} \lambda^h\right)$$

and holes will be excited in the low energy region $[-B, B]$. Solutions λ_α exist therefore only in the regions $(-\infty, -B]$ and $[B, \infty)$ and we are led to the magnetization equation,

$$\sigma_B(\lambda) + \left(\int_{-\infty}^{-B} + \int_B^\infty \right) K(\lambda - \lambda') \sigma_B(\lambda') d\lambda' = f_{GN}(\lambda),$$

with

$$f_{GN}(\lambda) = \frac{N^+}{\pi} \frac{(c/2)}{(c/2)^2 + (\lambda - 1)^2} + \frac{N^-}{\pi} \frac{(c/2)}{(c/2)^2 + (\lambda + 1)^2}.$$

To summarize, the form of the magnetization equation depends on the spectrum of excitations. While for the Kondo model we have only one depletion region, $(-\infty, B]$ and only *one* magnetic Fermi point in the Kondo model, there are two points in the other models due to the symmetric form of their excitation spectrum.

In his paper A. Zvyagin has introduced a second depletion region and second Fermi point to the Kondo magnetization equation. The state thus constructed has excitations in the range $[B, \infty)$ with each hole having energy of the order of the cutoff. The state is therefore infinitely excited and has no meaning.

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